

# Exposed positive maps in $M_4(\mathbb{C})$

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## Abstract

It is shown that the family of so called Breuer-Hall maps in  $M_4(\mathbb{C})$  possesses *strong spanning property* and hence they are exposed in the convex cone of positive maps in  $M_4(\mathbb{C})$ .

## 1 Introduction

Recall that a linear map  $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$  is positive if it maps a cone of positive elements in  $\mathcal{B}(\mathcal{K})$  into a cone of positive elements in  $\mathcal{B}(\mathcal{H})$ , that is,  $\Phi(\mathcal{B}_+(\mathcal{K})) \subset \mathcal{B}_+(\mathcal{H})$  [1]–[5]. Throughout this paper we use the standard notation:  $\mathcal{B}(\mathcal{H})$  denotes a  $\mathbb{C}^*$ -algebra of bounded operators in  $\mathcal{H}$  and  $\mathcal{B}_+(\mathcal{H})$  denotes a convex cone of positive operators in  $\mathcal{B}(\mathcal{H})$  (recall that  $a \in \mathcal{B}_+(\mathcal{H})$  iff  $a = bb^*$  for some  $b \in \mathcal{B}(\mathcal{H})$ ).

Let  $\mathcal{P}$  denotes a convex cone of positive maps  $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$  and let  $\mathcal{P}^\circ$  denote a dual cone [6, 7, 8]

$$\mathcal{P}^\circ = \text{conv} \{ P_x \otimes P_y ; \langle y | \Phi(P_x) | y \rangle \geq 0, \Phi \in \mathcal{P} \}, \quad (1)$$

where  $P_x = |x\rangle\langle x|$  and  $P_y = |y\rangle\langle y|$ . It is clear that  $\mathcal{P}^{\circ\circ} = \mathcal{P}$ , that is, one may consider  $\mathcal{P}$  as a dual cone to the convex cone of separable operators in  $\mathcal{H} \otimes \mathcal{K}$ . Recall that a face of  $\mathcal{P}$  is a convex subset  $F \subset \mathcal{P}$  such that if the convex combination  $\Phi = \lambda\Phi_1 + (1-\lambda)\Phi_2$  of  $\Phi_1, \Phi_2 \in \mathcal{P}$  belongs to  $F$ , then both  $\Phi_1, \Phi_2 \in F$ . If a ray  $\{\lambda\Phi : \lambda > 0\}$  is a face of  $\mathcal{P}$  then it is called an extreme ray, and we say that  $\Phi$  generates an extreme ray. For simplicity we call such  $\Phi$  an extremal positive map. A face  $F$  is exposed if there exists a supporting hyperplane  $H$  for a convex cone  $\mathcal{P}$  such that  $F = H \cap \mathcal{P}$ . The property of ‘being an exposed face’ may be reformulated as follows: A face  $F$  of  $\mathcal{P}$  is exposed iff there exists  $a \in \mathcal{B}_+(\mathcal{H})$  and  $|h\rangle \in \mathcal{H}$  such that

$$F = \{ \Phi \in \mathcal{P} ; \Phi(a)|h\rangle = 0 \}.$$

A positive map  $\Phi \in \mathcal{P}$  is exposed if it generates 1-dimensional exposed face. Let us denote by  $\text{Ext}(\mathcal{P})$  the set of extremal points and  $\text{Exp}(\mathcal{P})$  the set of exposed points of  $\mathcal{P}$ . Due to Straszewicz theorem [8]  $\text{Exp}(\mathcal{P})$  is a dense subset of  $\text{Ext}(\mathcal{P})$ . Thus every extreme map is the limit of some sequence of exposed maps meaning that each entangled state may be detected by some exposed positive map. Hence, a knowledge of exposed maps is crucial for the full characterization of separable/entangled states of bi-partite quantum systems. For recent papers on exposed maps see e.g. [9, 10, 11, 12, 13, 14].

Now, if  $F$  is a face of  $\mathcal{P}$  then

$$F' = \text{conv}\{ a \otimes |h\rangle\langle h| \in \mathcal{P}^\circ : \Phi(a)|h\rangle = 0, \Phi \in F \} . \quad (2)$$

defines a face of  $\mathcal{P}^\circ$  (one calls  $F'$  a dual face of  $F$ ). Actually,  $F'$  is an exposed face. One proves [6] the following

**Proposition 1.1** *A face  $F$  is exposed iff  $F'' = F$ .*

In this paper we prove the exposedness of a class of positive maps in  $M_4(\mathbb{C})$  using a different tool based on the *strong spanning property*.

## 2 Exposed maps vs. strong spanning property

A linear map  $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$  is called *irreducible* if  $[\Phi(X), Z] = 0$  for all  $X \in \mathcal{B}(\mathcal{K})$  implies  $Z = \lambda \mathbb{I}_{\mathcal{H}}$ . In a recent paper [10] (see also [11] and [5] in connection to unextendible positive maps) we proved the following

**Theorem 2.1** *Let  $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$  be a positive, unital, irreducible map, and let*

$$N_\Phi = \text{span}\{ a \otimes |h\rangle \in \mathcal{B}_+(\mathcal{K}) \otimes \mathcal{H} : \Phi(a)|h\rangle = 0 \} .$$

*If the subspace  $N_\Phi \subset \mathcal{B}(\mathcal{K}) \otimes \mathcal{H}$  satisfies*

$$\dim N_\Phi = (d_{\mathcal{K}}^2 - 1)d_{\mathcal{H}} , \quad (3)$$

*then  $\Phi$  is exposed.*

This theorem provides an analog of well known result [15] concerning optimality of positive maps (recall that a positive map  $\Phi$  is optimal if  $\Phi - \Lambda_{CP}$  is no longer positive for any completely positive map  $\Lambda_{CP}$ )

**Theorem 2.2** *Let  $\Phi : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$  be a positive map, and let*

$$M_\Phi = \text{span}\{ |x\rangle \otimes |h\rangle \in \mathcal{K} \otimes \mathcal{H} : \Phi(P_x)|h\rangle = 0 \} .$$

*If  $M_\Phi = \mathcal{K} \otimes \mathcal{H}$  or equivalently*

$$\dim M_\Phi = d_{\mathcal{K}}d_{\mathcal{H}} , \quad (4)$$

*then  $\Phi$  is optimal.*

This theorem was used recently to prove the optimality of generalized Choi maps [16, 17, 18]. In analogy to (4) we proposed to call (3) *strong spanning property*. Hence, as the spanning property is sufficient for optimality the *strong spanning property* is sufficient for exposedness.

To illustrate the Theorem 2.1 let us consider two simple examples for  $\mathcal{K} = \mathcal{H} = \mathbb{C}^2$

**Example 2.1 (Transposition map)** Let  $\tau : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  denotes the transposition  $\tau(X) = X^t$  in the standard basis  $\{e_1, e_2\}$  in  $\mathbb{C}^2$ . It is clear that  $\tau$  is irreducible and unital. One has  $\tau(P_x)|y\rangle = 0$  iff  $\langle \bar{x}|y\rangle = 0$ . The following 6 vectors  $|x_k\rangle \otimes |\bar{x}_k\rangle \otimes |y_k\rangle$  with

$$\begin{aligned} x_1 &= e_1, & y_1 &= e_2, \\ x_2 &= e_2, & y_2 &= e_1, \\ x_3 &= e_1 + e_2, & y_3 &= e_1 - e_2, \\ x_4 &= e_1 - e_2, & y_4 &= e_1 + e_2, \\ x_5 &= e_1 + ie_2, & y_5 &= e_1 + ie_2, \\ x_6 &= e_1 - ie_2, & y_6 &= e_1 - ie_2, \end{aligned}$$

are linearly independent in  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ . It shows that  $\tau$  is an exposed map.

**Example 2.2 (Reduction map)** Let  $R_2 : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  denotes the reduction map

$$R_2(X) = \mathbb{I}_2 \operatorname{Tr} X - X. \quad (5)$$

Again, one easily shows that  $R_2$  is unital and irreducible. Note that  $R_2(P_x)|y\rangle = 0$  iff  $|x\rangle = \lambda|y\rangle$  with  $\lambda \in \mathbb{C}$ . Taking

$$x_1 = e_1, x_2 = e_2, x_3 = e_1 + e_2, x_4 = e_1 - e_2, x_5 = e_1 + ie_2, x_6 = e_1 - ie_2, \quad (6)$$

one shows that 6 vectors  $|x_k\rangle \otimes |\bar{x}_k\rangle \otimes |x_k\rangle$  are linearly independent in  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ . It proves that  $R_2$  is an exposed map. Note that  $R_n : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  defined by

$$R_n(X) = \mathbb{I}_n \operatorname{Tr} X - X, \quad (7)$$

is no longer exposed for  $n > 2$  since it is not extremal.

### 3 Breuer-Hall map in $M_4(\mathbb{C})$

Consider now the Robertson map [19] defined by

$$\Phi_0(X) = \frac{1}{2} \left( R_4(X) - U_0 X^t U_0^\dagger \right) = \frac{1}{2} \left( \mathbb{I}_4 \operatorname{Tr} X - X - U_0 X^t U_0^\dagger \right), \quad (8)$$

where  $U$  is a unitary antisymmetric matrix given by

$$U_0 = \left( \begin{array}{c|c} i\sigma_y & \mathbb{O}_2 \\ \hline \mathbb{O}_2 & i\sigma_y \end{array} \right). \quad (9)$$

The normalization factor ‘1/2’ guaranties that  $\Phi_0(\mathbb{I}_4) = \mathbb{I}_4$ . It was shown that  $\Phi_0$  is an extremal indecomposable map. One has

$$\Phi_0(P_x) = \frac{1}{2} \left( \mathbb{I}_4 - P_x - P_{U\bar{x}} \right), \quad (10)$$

and  $P_x$  and  $P_{U\bar{x}}$  are mutually orthogonal projectors for an arbitrary normalized  $x \in \mathbb{C}^4$ . Therefore  $\Phi_0(P_x)|y\rangle = 0$  iff  $|y\rangle = |x\rangle$  or  $|y\rangle = U|\bar{x}\rangle$ .

**Proposition 3.1** *The map  $\Phi_0$  is irreducible.*

**Proof:** let us observe that  $\Phi_0(X)$  may be rewritten as follows [20]

$$\Phi_0(X) = \left( \frac{\mathbb{I}_2 \text{Tr } X_{22}}{-[X_{21} + R_2(X_{12})]} \middle| \frac{-[X_{12} + R_2(X_{21})]}{\mathbb{I}_2 \text{Tr } X_{11}} \right), \quad (11)$$

where  $X = \sum_{i,j} e_{ij} \otimes X_{ij}$  and  $e_{ij}$  are matrix units in  $M_2(\mathbb{C})$ . Suppose now that there exists  $Z \in M_4(\mathbb{C})$  such that  $[\Phi_0(X), Z] = 0$  for all  $X \in M_4(\mathbb{C})$ . Denoting  $Y = \Phi_0(X)$  one has

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \cdot \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \cdot \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

implies

$$Y_{12}Z_{21} = Z_{12}Y_{21}, \quad (12)$$

$$Y_{21}Z_{12} = Z_{21}Y_{12}, \quad (13)$$

$$Y_{21}Z_{11} + Y_{22}Z_{21} = Z_{21}Y_{11} + Z_{22}Y_{21}, \quad (14)$$

$$Y_{11}Z_{12} + Y_{12}Z_{22} = Z_{11}Y_{12} + Z_{12}Y_{22}. \quad (15)$$

Note, that if  $X_{12} = X_{21} = 0$ , then necessarily  $Z_{12} = Z_{21} = 0$  and hence  $Z$  is block-diagonal and hence equations (14) and (15) reduce to

$$Y_{21}Z_{11} = Z_{22}Y_{21},$$

$$Y_{12}Z_{11} = Z_{22}Y_{12}.$$

Taking  $X_{12} = a\mathbb{I}_2$  and  $X_{21} = b\mathbb{I}_2$  with  $a, b \in \mathbb{C}$ , one gets  $Y_{12} = Y_{21} = -(a+b)\mathbb{I}_2$  and hence  $Z_{11} = Z_{22} =: Z_0$ . Finally, one obtains the following condition for the diagonal block  $Z_0$ :

$$[X_{12} - X_{21}, Z_0] = 0,$$

and since  $X_{12}$  and  $X_{21}$  are arbitrary, it implies  $Z_0 = c\mathbb{I}_2$  and hence  $Z = c\mathbb{I}_4$ , which ends the proof of irreducibility of  $\Phi_0$ .  $\square$

Simple computer algebra enables one to prove the following

**Proposition 3.2** *The following 60 vectors  $|x_\alpha\rangle \otimes |\bar{x}_\alpha\rangle \otimes |y_\alpha\rangle$  with  $|x_\alpha\rangle$  belonging to*

$$\begin{aligned} x_k &= e_k, \quad (k = 1, 2, 3, 4) \\ x_{kl}^+ &= e_k + e_l, \quad (k < l) \\ x_{kl}^- &= e_k - e_l, \quad (k < l), (k, l) \neq (1, 2), \quad (k, l) \neq (3, 4) \\ \tilde{x}_{kl}^+ &= e_k + ie_l, \quad (k < l) \\ \tilde{x}_{kl}^- &= e_k - ie_l, \quad (k < l), (k, l) \neq (1, 2), \quad (k, l) \neq (3, 4) \\ x'_1 &= e_1 + e_2 + e_3 \\ x'_2 &= ie_1 + e_2 + e_3 \\ x'_3 &= e_1 + ie_2 + e_3 \\ x'_4 &= e_2 + e_3 + e_4 \\ x'_5 &= e_2 + ie_3 + e_4 \\ x'_6 &= e_2 + e_3 + ie_4 \end{aligned}$$

and  $|y_\alpha\rangle = |x_\alpha\rangle$  or  $|y_\alpha\rangle = U_0|\bar{x}_\alpha\rangle$  are linearly independent in  $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ .

This way we have proved

**Theorem 3.1** *The Robertson map  $\Phi_0$  is exposed.*

Consider now the family of so called Breuer-Hall map in  $M_4(\mathbb{C})$

$$\Phi_U(X) = \frac{1}{2} \left( R_4(X) - UX^t U^\dagger \right) = \frac{1}{2} \left( \mathbb{I}_4 \text{Tr } X - X - UX^t U^\dagger \right), \quad (16)$$

where  $U$  is an arbitrary unitary antisymmetric matrix. Note, that the above formula defines unital positive map in  $M_n(\mathbb{C})$  ( $n$  even) if  $U$  is an antisymmetric unitary from  $M_n(\mathbb{C})$  and we change the normalization factor  $2^{-1} \rightarrow (n-2)^{-1}$ . Clearly, for  $U = U_0$  it reproduces the original Robertson map. It was shown [21, 22] that  $\Phi_U$  is positive and indecomposable. Moreover, it turns out [21] that  $\Phi_U$  is nd-optimal, that is,  $\Phi_U - \Lambda_D$  is no longer a positive map for an arbitrary decomposable map  $\Lambda_D$ .

**Theorem 3.2** *The map  $\Phi_U$  is exposed.*

**Proof:** let us observe that  $\Phi_U(\mathbb{I}_4) = \mathbb{I}_4$ . Note that if the family of vectors

$$|x_\alpha\rangle \otimes |\bar{x}_\alpha\rangle \otimes |x_\alpha\rangle, \quad |x_\alpha\rangle \otimes |\bar{x}_\alpha\rangle \otimes U_0|\bar{x}_\alpha\rangle, \quad (17)$$

spans  $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ , then for arbitrary unitary operator  $\mathbb{V} : \mathbb{C}^{4 \otimes 3} \rightarrow \mathbb{C}^{4 \otimes 3}$  the family of vectors

$$\mathbb{V}(|x_\alpha\rangle \otimes |\bar{x}_\alpha\rangle \otimes |x_\alpha\rangle), \quad \mathbb{V}(|x_\alpha\rangle \otimes |\bar{x}_\alpha\rangle \otimes U_0|\bar{x}_\alpha\rangle), \quad (18)$$

spans  $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$  as well. Taking  $\mathbb{V} = V \otimes \bar{V} \otimes V$ , where  $V : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  is a unitary operator, one finds that the following vectors

$$|Vx_\alpha\rangle \otimes |\bar{Vx}_\alpha\rangle \otimes |Vx_\alpha\rangle, \quad |Vx_\alpha\rangle \otimes |\bar{Vx}_\alpha\rangle \otimes U|\bar{Vx}_\alpha\rangle, \quad (19)$$

with

$$U = VU_0V^t, \quad (20)$$

span  $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ . Note, that  $VU_0V^t$  defines antisymmetric unitary for an arbitrary unitary  $V$ . Moreover, any antisymmetric unitary  $U$  may be written via (20) for an appropriate unitary  $V$ , that is,  $U_0$  is a canonical form for an antisymmetric unitary matrix in  $M_4(\mathbb{C})$ . Indeed, note that for any antisymmetric unitary  $U$  if  $\lambda$  is an eigenvalue so is  $-\lambda$  and  $\lambda = e^{i\alpha}$ . It is, therefore, clear that

$$U = R \text{diag}\{e^{i\alpha_1}i\sigma_y, e^{i\alpha_2}i\sigma_y\} R^t, \quad (21)$$

where  $R \in M_4(\mathbb{R})$  is an orthogonal matrix. It proves that  $\Phi_U$  possesses the strong spanning property for an arbitrary antisymmetric unitary  $U$ . It remains to show that  $\Phi_U$  is irreducible. We have already proved that  $[\Phi_0(X), Z] = 0$  for all  $X \in M_4(\mathbb{C})$  implies  $Z \sim \mathbb{I}_4$ . Note, that

$$0 = [\Phi_0(X), Z] = V^\dagger [V\Phi_0(X)V^\dagger, VZV^\dagger]V = V^\dagger [\Phi_U(X'), Z']V,$$

where  $U = VU_0V^t$ ,  $X' = V^\dagger XV$  and  $Z' = VZV^\dagger$ . Now,  $[\Phi_U(X'), Z'] = 0$  for all  $X' \in M_4(\mathbb{C})$  implies  $Z' \sim \mathbb{I}_4$  which proves irreducibility of  $\Phi_U$ .

Hence,  $\Phi_U$  possesses the *strong spanning property* for an arbitrary  $U$  and being irreducible it is an exposed map.  $\square$

## 4 Conclusions

We have shown that so called Breuer-Hall positive maps [21, 22] in  $M_4(\mathbb{C})$  satisfies the *strong spanning property* and hence they are exposed in the convex cone of positive maps in  $M_4(\mathbb{C})$ . Therefore, this class defines the most efficient tool for detecting quantum entanglement (any entangled state may be detected by some exposed map (entanglement witness)). In a recent paper [14] it was shown that Breuer-Hall maps in  $M_n(\mathbb{C})$  (and  $n$  even) are exposed for an arbitrary antisymmetric unitary  $U$ . Interestingly, numerical analysis shows that for  $n > 4$  these maps do not possess the *strong spanning property*. Hence, for  $n > 4$  they provide an analog of the Choi map which is known to be optimal (even extremal) but does not satisfy spanning property (4). Actually, numerical analysis shows that

$$D_n := \dim N_\Phi = \frac{1}{6}n(n+1)(5n-2) .$$

Note that

$$D_n \leq n(n^2 - 1) ,$$

and the equality holds only for  $n = 4$ .

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